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Notes on the $X = M = K$ conjecture

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Abstract

This is an expanded version of a talk presented at the 2004 workshop on Combinatorial Aspects of Integrable Systems at the Research Institute of Mathematical Sciences, Kyoto, Japan.

We show that in the large rank limit, for any nonexceptional affine family of root systems, the one-dimensional sums associated to tensor products of Kirillov-Reshetikhin modules, have a very simple relationship to those of type A.

1 $X = M$ conjecture

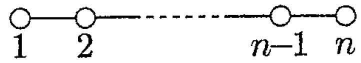
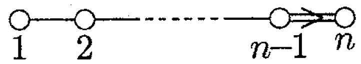
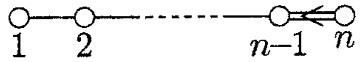
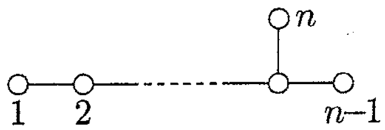
1.1 Notation

Let $\mathfrak{g} \supset \mathfrak{g}' \supset \overline{\mathfrak{g}}$ where $\mathfrak{g}$ is a Lie algebra of nonexceptional affine type, $\mathfrak{g}'$ is its derived subalgebra and $\overline{\mathfrak{g}}$ its canonical simple Lie subalgebra. Let I and $I \backslash \{0\}$ be the vertex sets of the Dynkin diagrams of $\mathfrak{g}$ and $\overline{\mathfrak{g}}$ respectively, where $0 \in I$ is the distinguished node [9]. Let $U_q(\mathfrak{g}) \supset U_q'(\mathfrak{g}) \supset U_q(\overline{\mathfrak{g}})$ be the quantized universal enveloping algebras associated to $\mathfrak{g} \supset \mathfrak{g}' \supset \overline{\mathfrak{g}}$ respectively [10]. Let $\{\omega_i \mid i \in I \backslash \{0\}\}$ be the fundamental weights of $\overline{\mathfrak{g}}$ and $P^+(\overline{\mathfrak{g}}) = \bigoplus_{i \in I \backslash \{0\}} \mathbb{Z}_{\geq 0}\, \omega_i$ the dominant weights. For $\lambda \in P^+(\overline{\mathfrak{g}})$ denote by V^λ the irreducible $U_q(\overline{\mathfrak{g}})$-module of highest weight λ. For $r \in I \backslash \{0\}$ and $s \in \mathbb{Z}_{>0}$ let $W_s^{(r)}$ be the finite-dimensional $U_q'(\mathfrak{g})$-module known as the Kirillov-Reshetikhin (KR) module [8] [7]. Conjecturally $W_s^{(r)}$ is irreducible and has an affine crystal base $B^{r,s}$.

We warn the reader that unless otherwise stated, we use the opposite of Kashiwara's tensor product convention for crystal graphs.

We shall use an encoding of dominant weights $\lambda = \sum_{i \in I \backslash \{0\}} a_i \omega_i$ by partitions. Using the Dynkin diagram labeling given in (1.1), and assuming that $a_i = 0$ for i a spin node (that is, $i = n$ for types B_n, C_n, D_n and $i = n-1$

for D_n) we identify the weight λ with the partition that has a_i columns of height i for all i.

$\overline{\mathfrak{g}}_n$	Dynkin diagram
A_n	
B_n	
C_n	
D_n	

1.2 Classical decomposition of KR modules

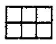
$W_s^{(r)}$ is generally reducible as a $U_q(\overline{\mathfrak{g}})$-module. This decomposition, prescribed by [8] [7], has the form

$$W_s^{(r)} \cong V^{s\omega_r} \oplus \text{“children”}$$

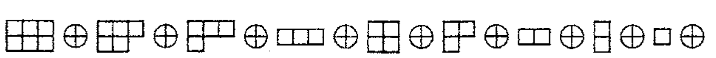
We give examples of this decomposition below, using the encoding of a weight as a partition. Thus $s\omega_r$ is the rectangle with r rows and s columns. According to [8] [7] the “children” are obtained by removing certain kinds of “tiles” from this rectangle.

Example 1. Let $r = 2$, $s = 3$, and the rank of $\overline{\mathfrak{g}}$ large. We use the labeling of affine Dynkin diagrams given in [8] [7]. The classical decompositions of $W_3^{(2)}$ are given as follows.

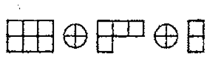
1. $\mathfrak{g} = A_n^{(1)}$: No children; Remove the empty tile $\varnothing$

   $W_3^{(2)} =$ 

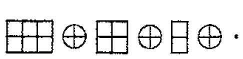
2. $\mathfrak{g} = D_{n+1}^{(2)}, A_{2n}^{(2)\dagger}$: Remove □

   $W_3^{(2)} =$ 

3. $\mathfrak{g} = C_n^{(1)}, A_{2n}^{(2)}$: Remove 

   $W_3^{(2)} =$ 

4. $\mathfrak{g} = B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}$: Remove 

   $W_3^{(2)} =$ 

1.3 The X formula

Consider any finite tensor product of KR modules

$$W^L := \bigotimes_{r,s} (W_s^{(r)})^{\otimes L_s^{(r)}}.$$

W^L has a $U_q(\overline{\mathfrak{g}})$-equivariant grading by the energy function D [12] [8] [7]. For $\lambda \in P^+(\overline{\mathfrak{g}})$, the one-dimensional sum $X_{L,\lambda}(t)$ is by definition the graded multiplicity of V^λ in the restriction of W^L to $U_q(\overline{\mathfrak{g}})$. It can be described entirely in terms of the combinatorics of the crystal graph of W^L. This definition makes sense in the cases where the KR modules that occur in W^L and their crystal bases have been constructed.

1.4 X = M

Let $M_{L,\lambda}(t)$ be the fermionic formula [8] [7]. It is defined for all L representing finite tensor products of KR modules and all $\lambda \in P(\overline{\mathfrak{g}})^+$. It is conjectured there that $X_{L,\lambda}(t) = M_{L,\lambda}(t)$.

2 The K formula

We now consider the one-dimensional sums for nonexceptional affine algebras in the special case that the rank is large with respect to the data (L, λ). We have observed that the one-dimensional sums are well-behaved and have conjectured that they have a simple formula in terms of the one-dimensional sums of type A. This conjectural formula is called the K formula.

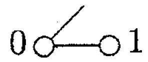
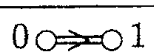
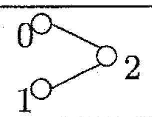
2.1 The large rank limit

Let $\{\mathfrak{g}_n\}$ be nonexceptional family of affine algebras where $\overline{\mathfrak{g}}_n$ has rank n. To obtain the Dynkin diagram of a nonexceptional affine algebra one attaches the 0 node somewhere on the left end of one of the classical Dynkin diagrams in (1.1). By examining the fermionic formulas one may prove the following result.

Proposition 2. *[20] For each nonexceptional family $\mathcal{F} = \{\mathfrak{g}_n\}$ of affine algebras, there is a well-defined large rank limit*

$$M_{L,\lambda}^{\mathcal{F}}(t) = \lim_{n\to\infty} M_{L,\lambda}^{\mathfrak{g}_n}(t)$$

called the stable fermionic formula. There are 4 distinct families of $M^{\mathcal{F}}$, *labeled by* $\diamondsuit \in \{\varnothing, \square, \square\!\square, \begin{smallmatrix}\square\\\square\end{smallmatrix}\}$. *This grouping of affine diagrams depends on the way in which the zero node is attached.*

family	affine root systems	attachment of 0 node
$M^{\varnothing}$	$A_n^{(1)}$	
$M^{\square}$	$A_{2n}^{(2)}, D_{n+1}^{(2)}$	
$M^{\square\!\square}$	$C_n^{(1)}, A_{2n}^{(2)\dagger}$	
$M^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$	$B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}$	

The grouping is the same as that for the classical decomposition of KR modules $W_s^{(r)}$; see section 1.2.

2.2 Ubiquity of type A

Kleber [10] observed that the character $Q_s^{(r)}$ of the KR module $W_s^{(r)}$ for r a nonspin node, should behave like such a character of type $A_n^{(1)}$. The Q-system [8] [7] is a relation among the KR characters, giving an expression for $(Q_s^{(r)})^2 - Q_{s-1}^{(r)}Q_{s+1}^{(r)}$ in terms of $Q_{s'}^{(i)}$ for i adjacent to r in the Dynkin diagram of $\overline{\mathfrak{g}}$. For a fixed r and in large rank, near r the Dynkin diagram always looks locally like that of type A_n. Therefore such KR characters have the same relations as those in type A.

2.3 Minimal affinizations

For $\tau \in P^+(\overline{\mathfrak{g}})$ (containing no spin weight) let W^τ be the associated minimal affinization [4]. It is an irreducible finite-dimensional $U_q'(\mathfrak{g})$-module with $U_q(\overline{\mathfrak{g}})$-decomposition of the form

$$W^\tau \cong V^\tau \oplus \text{children}.$$

Conjecture 3. *For L containing no spin weights, up to filtration, as $U_q'(\mathfrak{g})$-modules,*

$$W^L \cong \bigoplus_\tau X^A_{L,\tau}(1)\, W^\tau.$$

In type A it is known that $W^\tau \cong V^\tau$. This agrees with the definition of $X_{L,\lambda}$ as a graded tensor product branching multiplicity.

2.4 Decomposition of minimal affinizations

Define the branching coefficients $b_{\tau\lambda} \in \mathbb{Z}_{\geq 0}$ by the $U_q(\overline{\mathfrak{g}})$-decomposition

$$W^\tau \cong \bigoplus_\lambda b_{\tau\lambda} V^\lambda.$$

Let $P_\diamondsuit$ be the set of partitions that are tiled by $\diamondsuit$. Explicitly, $P_\varnothing$ is the singleton set containing just the empty partition, $P_\square$ is the set of all partitions, $P_{\square\square}$ is the set of partitions with even row lengths, and $P_{\substack{\square\\\square}}$ is the set of partitions with even column lengths.

Conjecture 4. *[3] If $\mathfrak{g}$ is in the family $\diamondsuit$ and τ contains no spin weights then*

$$b_{\tau\lambda} = \sum_{\mu \in P_\diamondsuit} c^\tau_{\lambda\mu}$$

where $c^\tau_{\lambda\mu}$ is the Littlewood-Richardson coefficient or type A_n tensor product multiplicity defined by

$$V^\lambda \otimes V^\mu \cong \oplus_\tau c^\tau_{\lambda\mu} V^\tau$$

This conjecture has been proved by Chari for many cases of τ of the form $s\omega_r$ [2].

2.5 The K-formula

Comparing the decomposition of W^L directly into V^λ or via W^τ, we have

$$X^\diamondsuit_{L,\lambda}(1) = \sum_\tau X^A_{L,\tau}(1) \sum_{\mu \in P_\diamondsuit} c^\tau_{\lambda\mu}.$$

Inserting a t strategically, we define the K formula

$$K^\diamondsuit_{L,\lambda}(t) = t^{|L|-|\lambda|} \sum_\tau X^A_{L,\tau}(t^2) \sum_{\mu \in P_\diamondsuit} c^\tau_{\lambda\mu}.$$

Originally the K formula was discovered through t-analogues of creation operators for the symmetric functions given by the large rank limits of classical characters [20].

2.6 $\mathbf{X} = \mathbf{M} = \mathbf{K}$

Conjecture 5. *[20] For large rank, $X = M = K$.*

This was previously known for $\diamondsuit = \varnothing$ (type $A_n^{(1)}$) and L general [13]. A special case has independently been conjectured by Lecouvey [14].

Our main result is:

Theorem 6.
$$X^{\diamondsuit} = K^{\diamondsuit}$$
for $\diamondsuit \in \{\square, \square\square\}$ *and* L *consisting of tensor factors of the form* $W_s^{(1)}$.

The proof is to show bijectively that $X^{\diamondsuit}$ satisfies the definition of $K^{\diamondsuit}$.

3 Recording tableaux and the X formula

3.1 Crystals $B^{1,1}$

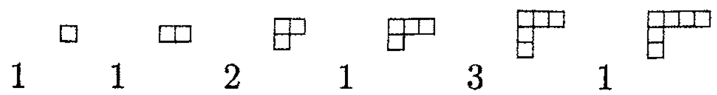
For simplicity we consider the case that W^L is a tensor power of the KR crystal $B^{1,1}$. In this context we will just regard L as the single integer previously denoted $L_1^{(1)}$. We also assume that the rank n of the classical subalgebra $\overline{\mathfrak{g}}$ is large, say, $n > L$. For each partition $\diamondsuit \in \{\varnothing, \square, \square\square, \boxminus\}$, we fix a representative affine family $X_N^{(r)}$. Its crystal $B_{\diamondsuit} := B^{1,1}$ is drawn in Figure 1.

3.2 Walks in Young's lattice

Let $b \in B_{\diamondsuit}^{\otimes L}$ be a classical highest weight vector (that is, a $U_q(\overline{\mathfrak{g}})$-highest weight vector). Let $\lambda^{(i)} \in P(\overline{\mathfrak{g}})^+$ be the weight of the first i steps in b. Identifying dominant weights with partitions, the path b can be encoded by the sequence of partitions $\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(L)}$. These sequences should be regarded as recording tableaux in the language of the Robinson-Schensted correspondence. The shape of such a tableau is by definition the ending partition.

We give examples below. Every unbarred (resp. barred) step r (resp. $\bar{r}$) adds (resp. removes) a cell to (resp. from) the r-th row in the partition diagram. A step $\varnothing$ leaves the partition unchanged.

1. A standard tableau of shape $(4, 1, 1)$ and type $\varnothing$-path:



(1)

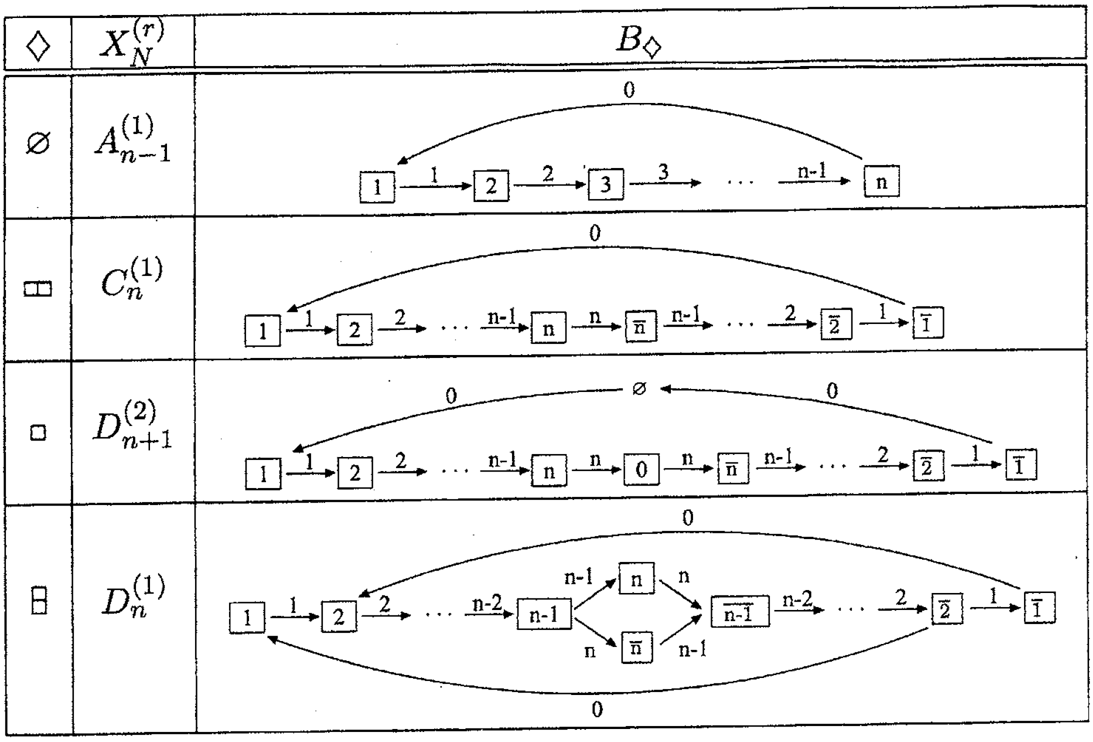
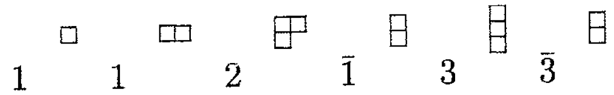


Figure 1: Representative affine families

Usually a standard tableau is written as follows, where the entry i is in the r-th row if and only if the i-th element in the path has value r.

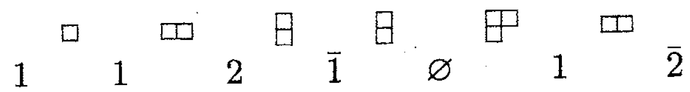
$$P = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 6 \\ \hline 3 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array} \tag{2}$$

2. An oscillating tableau of shape $(1,1)$ and type ▭ (or ⊟) path:



(3)

3. A Motzkin tableau of shape (2) and type □ path:



3.3 Energy function on paths

In general the way to compute the energy function D is given in [7]; see also [16]. Under the current assumptions the energy function can be computed as follows. We now reverse the order of the paths (the opposite of

Kashiwara's convention for crystal graphs). The subscripts label the gaps between steps in the path. The subscripts at positions that contribute to the energy function are underlined.

1. Type $\varnothing$: Sum the positions of descents (gaps between steps where the step on the left is greater than the one on the right).

$$\begin{aligned} c &= 1_1 3_{\underline{2}} 1_3 2_{\underline{4}} 1_5 1 \\ D_{\varnothing}(c) &= 2 + 4 = 6 \end{aligned} \tag{4}$$

2. Type $\boxbar$: same, with $1 < 2 < 3 < \cdots < \bar{3} < \bar{2} < \bar{1}$

$$\begin{aligned} b &= \bar{3}_{\underline{1}} 3_2 \bar{1}_{\underline{3}} 2_{\underline{4}} 1_5 1 \\ D_{\boxbar}(b) &= 1 + 3 + 4 = 8 \end{aligned} \tag{5}$$

3. Type $\boxminus$: same, except a descent of the form $\bar{1} > 1$ is counted twice.

4. Type $\square$:

   (a) Adjacent pairs $x > y$ and $\varnothing\varnothing$ count double

   (b) Pairs $x\varnothing$ and $\varnothing x$ count once for $x \neq \varnothing$

   (c) If the rightmost letter is $\varnothing$, there is a descent to its right.

$$\begin{aligned} b &= \bar{2}_{\underline{\underline{1}}} 1_{\underline{2}} \varnothing_{\underline{3}} \bar{1}_{\underline{\underline{4}}} 2_{\underline{\underline{5}}} 1_6 1_7 \\ D_{\square}(b) &= 1 \cdot 2 + 2 \cdot 1 + 3 \cdot 1 + 4 \cdot 2 + 5 \cdot 2 = 25. \end{aligned}$$

3.4 X formula

For $\diamondsuit \in \{\varnothing, \square, \boxbar, \boxminus\}$

$$X^{\diamondsuit}_{L,\lambda}(t) = \sum_{b \in P_{\diamondsuit}(L,\lambda)} t^{D_{\diamondsuit}(b)}$$

where $P_{\diamondsuit}(L, \lambda)$ is the set of classical highest weight vectors in $B_{\diamondsuit}^{\otimes L}$ of weight λ.

3.5 The required bijection

We shall only state it for the $\diamondsuit = \square\!\square$ case. To prove $X = K$ we must show that

$$X^{\square\!\square}_{L,\lambda}(t^2) = t^{L-|\lambda|} \sum_{\tau} X^{\varnothing}_{L,\tau}(t^2) \sum_{\substack{\mu \in \mathcal{P}^{\square\!\square} \\ |\mu| = L-|\lambda|}} c^{\tau}_{\lambda\mu} \tag{6}$$

Therefore we require a bijection

$$\begin{array}{ccc} \left\{\begin{matrix}\text{Oscillating}\\ \text{tableaux}\\ \text{shape } \lambda\\ \text{length } L\end{matrix}\right\} & \longrightarrow & \bigcup\limits_{\substack{|\tau|=L\\ \mu\in\mathcal{P}^{\square\!\square}}} \left\{\begin{matrix}\text{Standard}\\ \text{tableaux}\\ \text{shape } \tau\end{matrix}\right\} \times LR(\tau;\lambda,\mu) \\ b & \longmapsto & (c, Z) \end{array} \tag{7}$$

where Z is an element of a set $LR(\tau;\lambda,\mu)$ of cardinality $c^{\tau}_{\lambda\mu}$. We use the following realization [15]: $c^{\tau}_{\lambda\mu}$ is equal to the number of factorizations $T \cdot S$ of any fixed semistandard tableau P of shape τ, into semistandard tableaux T and S of shapes λ and μ respectively. The factorization is taken in the plactic monoid; see [6].

In summary, given the oscillating tableau b as input (see (3)), we must construct a standard tableau c, either in the form of a type $\varnothing$ path (see (1)) or in the more traditional form, denoted here as P (see (2)). Then we must also prescribe a factorization of P into $T \cdot S$ with T and S tableaux of shapes λ and μ respectively.

Moreover, the bijection must be grade-preserving:

$$2\,D_{\square\!\square}(b) = L - |\lambda| + 2\,D_{\varnothing}(c) \tag{8}$$

Example 7. Let b be as in (3); it has shape $\lambda = (1,1)$, $L = 6$, and by (5) it has $D(b) = 8$. We do not say how to find it yet, but the corresponding path c is given in (1); it has $D(c) = 6$ by (4). Then (8) becomes $2\cdot 8 = 6-2+2\cdot 6$.

4 The VXR map

We describe a way to compute the element c in the desired bijection (7). It uses the virtual crystal (VX) construction of [16] and the computation of the combinatorial R-matrix. Thus the name VXR.

4.1 Dual crystal

Let $B^\vee$ be the dual crystal graph of B [10]. By definition $B^\vee$ has a vertex $b^\vee$ for each $b \in B$. The arrows are reversed: $f_i(b^\vee) = c^\vee$ if and only if $f_i(c) = b$ for $b, c \in B$. For example, for $B_A = B^{1,1}$ of type $A_5^{(1)}$ we have (omitting zero arrows)

$$\begin{array}{ll} B_A: & \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{3} \boxed{4} \xrightarrow{4} \boxed{5} \xrightarrow{5} \boxed{6} \\ B_A^\vee: & \boxed{1^\vee} \xleftarrow{1} \boxed{2^\vee} \xleftarrow{2} \boxed{3^\vee} \xleftarrow{3} \boxed{4^\vee} \xleftarrow{4} \boxed{5^\vee} \xleftarrow{5} \boxed{6^\vee} \end{array}$$

For an element of $B_A^{\vee\otimes L}$, one may compute its energy using the usual rule for type $\varnothing$ paths, with the ordering $\cdots < 3^\vee < 2^\vee < 1^\vee$.

4.2 Virtual crystals

For certain affine root systems and KR crystals it was shown in [16] that crystals of nonsimply laced type could be realized using those of simply laced type. This is called the virtual crystal construction. For example, the KR crystal $B_C = B^{1,1}$ of type $C_n^{(1)}$, can be embedded into the tensor product $B_A \otimes B_A^\vee$ where $B_A = B^{1,1}$ is the KR crystal of type $A_{2n-1}^{(1)}$. Let us call this the virtual crystal (VX) embedding. Moreover the one-dimensional sum X can be entirely expressed in terms of the crystal of simply-laced type.

For example, define the embeddings Ψ and Ψ' by

$$\begin{array}{ccc} B_C & \xrightarrow{\Psi} & B_A^\vee \otimes B_A \\ i & \longrightarrow & (2n+1-i)^\vee \otimes i \\ \bar{i} & \longrightarrow & i^\vee \otimes (2n+1-i) \end{array} \tag{9}$$

$$\begin{array}{ccc} B_C & \xrightarrow{\Psi'} & B_A \otimes B_A^\vee \\ i & \longrightarrow & i \otimes (2n+1-i)^\vee \\ \bar{i} & \longrightarrow & (2n+1-i) \otimes i^\vee \end{array} \tag{10}$$

More generally, there is an embedding

$$\begin{gathered} B_C^{\otimes L} \xrightarrow{\Psi} (B_A^\vee \otimes B_A)^{\otimes L} \\ b_L \otimes \cdots \otimes b_1 \mapsto \Psi(b_L) \otimes \cdots \otimes \Psi(b_1) \end{gathered}$$

defined by the L-fold tensor product of the map (9). A similar construction can be made for Ψ'.

4.3 The R-matrix

If B and B' are the crystal bases of irreducible $U'_q(\mathfrak{g})$-modules then there is a unique isomorphism of affine crystal graphs $R_{B,B'} : B \otimes B' \to B' \otimes B$ called the combinatorial R-matrix [12].

We need an explicit computation of the following R-matrix. Here B_A has type $A^{(1)}_{2n-1}$.

$$\begin{array}{ll} B_A \otimes B_A^\vee \overset{R}{\leftrightarrow} B_A^\vee \otimes B_A & \\ \quad i \otimes j^\vee \leftrightarrow j^\vee \otimes i & \text{if } i \neq j \\ \quad i \otimes i^\vee \leftrightarrow (i+1)^\vee \otimes i+1 & \text{if } i < 2n \\ 2n \otimes 2n^\vee \leftrightarrow 1^\vee \otimes 1 & \end{array}$$

In particular the following diagram commutes:

$$\begin{array}{ccc} B_C & \xrightarrow{VX} & B_A^\vee \otimes B_A \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle R} \\ B_C & \xrightarrow{VX'} & B_A \otimes B_A^\vee \end{array} \tag{11}$$

4.4 Local energy function

Given B and B' as above, there is a map $H = H_{B,B'} : B \otimes B' \to \mathbb{Z}$ called the local energy function. See [12] [7].

We also have the value of the local energy function $H : B_A^\vee \otimes B_A \to \mathbb{Z}$, given in this case by

$$H(x \otimes y) = \begin{cases} 1 & \text{if } x \otimes y = 1^\vee \otimes 1 \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Similarly the local energy function $H : B_A \otimes B_A^\vee \to \mathbb{Z}$ is given by

$$H(x \otimes y) = \begin{cases} 1 & \text{if } x \otimes y = 2n \otimes 2n^\vee \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

4.5 VXR map

We show how to compute the part of the bijection (7) that, given $b \in P_{\square\square}(L, \lambda)$, determines an element $c \in P_{\varnothing}(L, \tau)$, and why the grade-preserving property (8) holds. At this time we do not compute the Littlewood-Richardson data $T \cdot S$.

By (11) the following diagram commutes.

$$
\begin{tikzcd}
 & B_C^{\otimes L} \arrow[r, equal] \arrow[d, "\Psi"'] & B_C^{\otimes L} \arrow[d, "\Psi'"] & \\
B_A^{\vee\otimes L} \otimes B_A^{\otimes L} & (B_A^{\vee} \otimes B_A)^{\otimes L} \arrow[l, "R_+"] \arrow[r, "R_0"'] & (B_A \otimes B_A^{\vee})^{\otimes L} \arrow[r, "R_-"'] & B_A^{\otimes L} \otimes B_A^{\vee\otimes L}
\end{tikzcd}
\tag{14}
$$

The maps R_+, R_0, and R_- are compositions of R-matrices of the form given in section 4.3. Consider the composite map $R_+ \circ \Psi$. Say it maps $b \mapsto d \otimes c$. This $c \in B_A^{\otimes L}$ is the one in the desired bijection.

In the following example $n = 3$. The computation of R_+ on $\Psi(b)$ is shown below. Here we write $\check{1}$ instead of $1^{\vee}$, etc.

$$
\begin{array}{c|cccccccccccc}
b & \multicolumn{2}{c}{\bar{3}} & \multicolumn{2}{c}{3} & \multicolumn{2}{c}{\bar{1}} & \multicolumn{2}{c}{2} & \multicolumn{2}{c}{1} & \multicolumn{2}{c}{1} \\
\hline
\Psi(b) & \check{3} & 4 & \check{4} & 3 & \check{1} & 6 & \check{5} & 2 & \check{6} & 1 & \check{6} & 1 \\
 & \check{3} & \check{5} & 5 & \check{1} & 3 & \check{5} & 6 & \check{6} & 2 & \check{6} & 1 & 1 \\
 & \check{3} & \check{5} & \check{1} & 5 & \check{5} & 3 & \check{1} & 1 & \check{6} & 2 & 1 & 1 \\
 & \check{3} & \check{5} & \check{1} & \check{6} & 6 & \check{1} & 3 & \check{6} & 1 & 2 & 1 & 1 \\
 & \check{3} & \check{5} & \check{1} & \check{6} & \check{1} & 6 & \check{6} & 3 & 1 & 2 & 1 & 1 \\
d \otimes c & \check{3} & \check{5} & \check{1} & \check{6} & \check{1} & \check{1} & 1 & 3 & 1 & 2 & 1 & 1
\end{array}
$$

From the path c one obtains the standard tableau P; see (2). The element d tells how to make the factorization $P \equiv T \cdot S$. We prefer to compute this another way later.

We now prove (8). The main ingredient is the following result.

Theorem 8. *[16] The virtual crystal embedding Ψ respects energy.*

More precisely, in our situation this means that

$$2\, D_{\square\square}(b) = D_{\varnothing}(d \otimes c).$$

Let $R : B_A^{\vee\otimes L} \otimes B_A^{\otimes L} \to B_A^{\otimes L} \otimes B_A^{\vee\otimes L}$ be the R-matrix and $H : B_A^{\vee\otimes L} \otimes B_A^{\otimes L} \to \mathbb{Z}$ be the local energy function [12]. Let $d' \otimes c' = R(d \otimes c) \in B_A^{\otimes L} \otimes B_A^{\vee\otimes L}$. By the definition of D [7] [16] we have

$$D(d \otimes c) = D(c) + D(c') + H(d \otimes c).$$

To prove (8) it is enough to show that

$$D(c') = D(c) \tag{15}$$

$$H(d \otimes c) = L - |\lambda| = 2 \cdot (\text{the number of barred steps in } b). \tag{16}$$

To prove (15) we note that the computations of $d' \otimes c'$ and $d \otimes c$ are entirely parallel. By the commutativity of the diagram (14) we see that $d' \otimes c' = R_- \circ \Psi'(b)$. Since the pairs of factors in $\Psi(b)$ and in $\Psi'(b)$ are just reversed, we have the following computation:

b	$\bar{3}$		3		$\bar{1}$		2		1		1	
$\Psi'(b)$	4	$\check{3}$	3	$\check{4}$	6	$\check{1}$	2	$\check{5}$	1	$\check{6}$	1	$\check{6}$
	4	2	$\check{2}$	6	$\check{4}$	2	$\check{1}$	1	$\check{5}$	1	$\check{6}$	$\check{6}$
	4	2	6	$\check{2}$	2	$\check{4}$	6	$\check{6}$	1	$\check{5}$	$\check{6}$	$\check{6}$
	4	2	6	1	$\check{1}$	6	$\check{4}$	1	$\check{6}$	$\check{5}$	$\check{6}$	$\check{6}$
	4	2	6	1	6	$\check{1}$	1	$\check{4}$	$\check{6}$	$\check{5}$	$\check{6}$	$\check{6}$
$d' \otimes c'$	4	2	6	1	6	6	$\check{6}$	$\check{4}$	$\check{6}$	$\check{5}$	$\check{6}$	$\check{6}$

So $c' = \check{6}\check{4}\check{6}\check{5}\check{6}\check{6}$. Compare this with $c = 131211$. In general one can show that c and c' have descents in the same positions, which implies that they have the same energy, proving (15).

To prove (16), one must use the fact that the desired value of H, is obtained as the sum of local energy functions $H_{B_A^\vee \otimes B_A}$ evaluated at adjacent tensor factors which are exchanged by the R-matrices $B_A^\vee \otimes B_A \to B_A \otimes B_A^\vee$ which comprise the computation of the R-matrix $B_A^{\vee \otimes L} \otimes B_A^{\otimes L} \to B_A^{\otimes L} \otimes B_A^{\vee \otimes L}$ [7] [16]. The latter R matrix is given by $R_- \circ R_0 \circ R_+^{-1}$. By (12) the only contributions to the energy are given by places where one has an application of the local R matrix of the form $1^\vee \otimes 1 \mapsto 2n \otimes 2n^\vee$. By studying the above computations we see that such exchanges happen an even number of times, and that they occur in symmetric pairs, with one occurrence during R_+^{-1} and the other during R_-. One also sees that the number of times that such exchanges occur during R_+^{-1} is the number of barred elements in b. From these considerations (16) follows.

5 DDF bijection

We now take a completely different approach to the desired bijection (7).

5.1 DDF

The following bijection is due to Delest, Dulucq, and Favreau [5]. Let $[L] = \{1, 2, \ldots, L\}$ and $\binom{[L]}{k}$ be the collection of subsets of $[L]$ of cardinality k.

The DDF bijection is between the following sets.

$$\begin{array}{ccccc}
\begin{bmatrix}\text{Oscillating}\\ \text{tableaux}\\ \text{of shape } \lambda\\ \text{length } L\end{bmatrix} & \longrightarrow & \displaystyle\bigcup_{A\in\binom{[L]}{|\lambda|}} \begin{bmatrix}\text{standard}\\ \text{tableaux}\\ \text{of shape } \lambda\\ \text{alphabet } A\end{bmatrix} & \times & \begin{bmatrix}\text{Fixed point}\\ \text{free}\\ \text{involutions}\\ \text{on } [L]-A\end{bmatrix}\\
b & \mapsto & (T, I) & &
\end{array}$$

Let $b = b_1 b_2 \cdots b_L$. Start with T and I both empty. For i from 1 to L do:

(D1) If $b_i = r$ then adjoin i to T at row r.

(D2) If $b_i = \bar{r}$ then reverse Schensted row insert on T at row r, ejecting the value a, say. Add the pair (i, a) to I.

Example 9. Let $b = 112\bar{1}3\bar{3}$ as before.

i	0	1	2	3	4	5	6
b_i		1	1	2	$\bar{1}$	3	$\bar{3}$
T	$\cdot$	$\begin{array}{\|c\|}\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1&2\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline 3\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline 3\\\hline 5\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 5\\\hline\end{array}$
I	()	()	()	()	(24)	(24)	(24)(16)

The output is $T = \begin{array}{|c|}\hline 3\\\hline 5\\\hline\end{array}$ and $I = (24)(16) = \begin{pmatrix}6&4&2&1\\1&2&4&6\end{pmatrix}$

The DDF bijection may be extended to a bijection

$$\begin{array}{ccccc}
\begin{bmatrix}\text{Motzkin}\\ \text{tableaux}\\ \text{of shape } \lambda\\ \text{length } L\end{bmatrix} & \longrightarrow & \displaystyle\bigcup_{A\in\binom{[L]}{|\lambda|}} \begin{bmatrix}\text{standard}\\ \text{tableaux}\\ \text{of shape } \lambda\\ \text{alphabet } A\end{bmatrix} & \times & \begin{bmatrix}\text{Involutions}\\ \text{on } [L]-A\end{bmatrix}.\\
b & \mapsto & (T, I) & &
\end{array}$$

In the extended DDF bijection there is an additional rule.

(D3) If $b_i = \varnothing$ then add a fixed point (i) to I.

5.2 The Burge correspondence

The Burge correspondence [1] is a bijection

$$\begin{array}{ccc} \left\{\begin{array}{c}\text{Fixed point free}\\ \text{Involutions}\end{array}\right\} & \to & \left\{\begin{array}{c}\text{Standard}\\ \text{tableaux}\\ \text{with}\\ \text{even rows}\end{array}\right\} \\ I & \mapsto & S \end{array}$$

It can be obtained as the restriction of the following bijection.

$$\{\text{Involutions}\} \to \left\{\begin{array}{c}\text{Standard}\\ \text{tableaux}\end{array}\right\}$$

To compute $I \mapsto S$, write the involution I as a two-line permutation. Reverse the lower word and row insert it into the empty tableau, obtaining S.

Example 10. Let $I = (24)(16)$ in cycle notation. Then in two-line notation we have

$$I = \begin{pmatrix} 1 & 2 & 4 & 6 \\ 6 & 4 & 2 & 1 \end{pmatrix} \qquad (\varnothing \leftarrow 1246) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 6 \\ \hline\end{array} = S$$

Here is another example with $I = (24)(5)(17)$. We have

$$I = \begin{pmatrix} 1 & 2 & 4 & 5 & 7 \\ 7 & 4 & 2 & 5 & 1 \end{pmatrix} \qquad (\varnothing \leftarrow 15247) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 7 \\ \hline 5 \\ \cline{1-1}\end{array} = S$$

5.3 Insert DDF and Burge data

Let $P \equiv T \cdot S$ be the standard tableau obtained by the plactic product of T and S [6]. Then define c to be the corresponding type $\varnothing$ path. This gives us the desired data c and $P \equiv T \cdot S$ for (7).

Example 11.

$$T \cdot S = \begin{array}{|c|}\hline 3 \\ \hline 5 \\ \hline\end{array} \cdot \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 6 \\ \hline\end{array} \equiv \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 6 \\ \hline 3 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array} = P$$

$$c = 112131$$

Here c is written in the unreversed order.

5.4 Finishing the proof

One may show that the VXR map and DDF bijection give the same answer. The DDF formulation is a composition of bijections and is therefore bijective. The VXR map was shown to be grade-preserving. This completes the proof of $X = K$ in the special case that was explained here.

6 Closing remarks

6.1 Level-rank duality

The type A one-dimensional sums satisfy a graded level-rank duality [17] [18] [19]

$$K^{\varnothing}_{L,\lambda}(t) = t^{||L||} K^{\varnothing}_{L^t,\lambda^t}(t^{-1})$$

where λ^t indicates the transpose or conjugate partition of λ, L^t is obtained by transposing all rectangles in L, and $||L|| = \sum_{i,j\geq 1} \binom{r_{ij}(L)}{2}$ where $r_{ij}(L) = \sum_{r\geq i} \sum_{s\geq j} L^{(r)}_s$.

This implies the following identity, where $|L| = \sum_{r,s} rsL^{(r)}_s$:

$$K^{\diamondsuit^t}_{L^t,\lambda^t}(t) = t^{||L||+|L|-|\lambda|} K^{\diamondsuit}_{L,\lambda}(t^{-1})$$

The $X = M = K$ conjecture implies that in large rank, types B and C have one-dimensional sums which are given by polynomials whose terms occur in the reverse order from each other. Just from the definitions it is entirely unclear why the one-dimensional sums of types B and C should have any relationship with each other.

6.2 The missing case

Our methods don't seem to work at all for type ⊟. Even though types ◫ and ⊟ involve exactly the same kinds of paths, the energy functions for these two types behave rather differently.

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